Indian Statistical Institute, Bangalore
B. Math.(hons.), Third Year, First Semester
Probability-III

Mid Term Supplementary Examination Date : 10 October 2024
Maximum marks: 20 Time: 2:30 hours

Answer any 4, each question carries 5 marks.

Kindly ensure your writing is clear and if you are using any results please provide as many details as you can.

1. Let $X_i, i = 1, 2, 3$ be random variables on a probability space $(\Omega, \mathcal{F}, P)$. Consider the random equation (in $t \in \mathbb{R}$):
$$X_1(\omega)t^2 + X_2(\omega)t + X_3(\omega) = 0. \tag{1}$$
   (i) Show that
   $$A \equiv \{\omega \in \Omega : \text{Equation (1) has two distinct real roots}\} \in \mathcal{F}.$$
   (ii) Let $T_1(\omega)$ and $T_2(\omega)$ denote the two roots of (1) on A. Let
   $$f_i(\omega) = \begin{cases} T_i(\omega) & \text{on } A \\ 0 & \text{on } A^c \end{cases}, \quad i = 1, 2.$$
   Show that (f_1, f_2) is $\langle \mathcal{F}, B(\mathbb{R}^2)\rangle$-measurable.

2. (i) Let $(\Omega, \mathcal{F}_1, \mu)$ be a σ-finite measure space. Let $T : \Omega \to \mathbb{R}$ be $\langle \mathcal{F}, B(\mathbb{R})\rangle$-measurable. Show that the induced measure μT^{-1} need not be σ-finite.
   (ii) Let $(\Omega_i, \mathcal{F}_i)$ be measurable spaces for $i = 1, 2$ and let $T : \Omega_1 \to \Omega_2$ be $\langle \mathcal{F}_1, \mathcal{F}_2\rangle$-measurable. Show that any measure μ on $(\Omega_1, \mathcal{F}_1)$ is σ-finite if μT^{-1} is σ-finite on $(\Omega_2, \mathcal{F}_2)$.

3. (i) Let X be a nonnegative random variable. Show that
   $$1 + (E[X])^2 \leq E[1 + X^2] \leq 1 + E[X]^2.$$
   (ii) If $Y \geq 0$ and $p > 0$, then
   $$E(Y^p) = \int_0^\infty p y^{p-1} P(Y > y)\, dy.$$

4. Let $\{X_i\}_{i\geq 1}$ be a sequence of identically distributed random variables, and let $M_n = \max\{|X_j| : 1 \leq j \leq n\}$.

(i) If $E|X_1|^\alpha < \infty$ for some $\alpha \in (0, \infty)$, then show that

$$\frac{M_n}{n^{1/\alpha}} \to 0 \quad \text{w.p. 1.} \tag{2}$$

(ii) Show that if $\{X_i\}_{i\geq 1}$ are i.i.d. satisfying (2) for some $\alpha > 0$, then

$$E|X_1|^\alpha < \infty.$$

5. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, P)$. Let $R = R(\omega)$ be the radius of convergence of the power series $\sum_{n=1}^{\infty} X_n r^n$. Then

   (i) show that R is a tail random variable,

   (ii) show that if $E(\log|X_1|)^+ = \infty$, then $R = 0$ almost surely (w.p. 1). And if $E(\log|X_1|)^+ < \infty$, then $R \geq 1$ almost surely (w.p. 1).